Contemporary Status of the Three-Body Problem with Coulomb Interactions

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Since pioneering work of Faddeev [1], the three-body quantum scattering theory has become a powerful tool for the investigation of many different processes in various areas of physics. However, the question of how to incorporate long-range Coulomb forces into the three-body scattering formalism remains as major obstacle to its wider-spread application.

Also most stationary approaches, based on integral equations, for taking into account the Coulomb interactions have remained formal In the approach proposed by Noble [2], the three-body integral equations are rewritten so that all Coulomb potentials are included in what had been the 'unperturbed' Green's function. The latter is changed into the three-body Coulomb Green's function which then enters the kernels of the new integral equations (as well as those of auxiliary three-body equations for quantities which in the original formulation had been ordinary two-body T-operators). Thus all unpleasant features related to the Coulomb interactions are hidden in the unknown three-body Coulomb Green's function. The problem of calculating the latter is not any simpler than the initial problem of solving for the full three-body Green's function. Merkuriev's approach [3] is based on the same idea, except that there the Coulomb potentials are split by means of suitable cut-off functions into 'inner' and 'outer' parts, and only the latter are incorporated into the (formerly free) three-body Green's function. Not surprisingly, the kernels of the Faddeev-type integral equations for the Green's function for the cut-off Coulomb plus short-range potentials have similar compactness properties as those for short-range potentials alone and, thus, can be treated by conventional methods. But for the determination of the auxiliary Green's function containing the 'outer' Coulomb potential parts, again only formal integral equations have been proposed and shown to possess compact kernels provided that all Coulomb interactions are repulsive [3]. Their explicit solution appears to be very difficult, and has not been attempted. For completeness we mention that the uniqueness of the solutions of the differential Faddeev equations for Coulomb-like potentials has been proved in a special class of functions [3], assuming that all three particles have charges of equal sign (repulsive Coulomb potentials). But it should be kept in mind that the boundary condition to be imposed on the solutions of the differential equations used in [3] was not complete. Recently the missing part of the needed boundary conditions was derived in our papers [4, 5]. We found the leading asymptotic terms of the three-body Coulomb scattering wave function in the asymptotic domain where two particles are close together (in the continuum) and far away from the third particle.

An important practical result has been derived by Veselova [6]. When considering the Faddeev integral equations with screened Coulomb potentials at energies below the breakup threshold, she was able to obtain from the kernel that term which in the zero-screening limit yields the so-called two-particle or center-of-mass Coulomb singularity, in such a form that it could be inverted explicitly. Thus the modified three-body integral equations with compact kernels were obtained. But this inversion procedure was only shown to work for energies below the breakup threshold. At energies above that threshold, three-particle singularities appeared [6, 3] which nobody has succeeded to handle until now.

Because of the difficulties in deriving proper equations for the kernels of three-body transition operators which are valid for all energies and are well suited for practical calculations, it appears more promising to split the problem into several independent parts. A first step consists in developing integral equations for effective-two-body transition amplitudes which describe all possible binary processes, i.e., processes in which a projectile impinges on a two-particle bound state leading again to a two-body final state ((in-)elastic and rearrangement collisions, or so-called $2 \rightarrow 2$ reactions). The search for appropriate equations for breakup amplitudes describing $2 \rightarrow 3$ reactions, or for three-body equations for amplitudes describing $3 \rightarrow 3$ processes, is deferred to a later stage.

Such an approach was developed in [7, 8]. Starting from the Alt-Grassberger-Sandhas (AGS) integral equations for the three-body transition operators [9], they can be reduced exactly by means of the so-called quasiparticle approach to a set of coupled, multichannel, Lippmann-Schwinger-type equations for effective-two-body (i.e., binary) transition amplitudes. By using the screening method, this formulation allowed the isolation, and subsequent extraction, of the leading (in the limit of vanishing screening radius) Coulomb singularity which then could be inverted explicitly. After renormalization, the various screened binary amplitudes have been shown to coincide, in the zero-screening limit, with the corresponding amplitudes as resulting from Dollar's time-dependent theory, in particular for energies above the three-body threshold. The unique relation between amplitudes as defined in the time-dependent and stationary screening and renormalization approach could be established also for the breakup ($2 \rightarrow 3$) amplitudes, but only for the case of two charged and one neutral particles. Thus, for the latter case [3], from the mathematical point of view the screening and renormalization approach provides a proof of the compactness of the corresponding (three-body) Faddeev or AGS integral equations, in a special class of functions.

In spite of the success of the screening and renormalization approach, not only as a method for proving the existence of various quantities of interest but also as a practical computational tool, it appears highly desirable to investigate the effective-two-body AGS equations directly for unscreened Coulomb potentials. The question of compactness of the kernels occurring therein depends on the analytical properties of their constituents, which are the so-called 'effective potentials' and 'effective free propagators'. The latter are known to have only a pole singularity 'at the on-shell point' (besides the three-body cut). For the effective potentials, however, no thorough investigation of their singularities has been performed up to now.

The aim of our work is to overcome that deficiency. We investigated the analytical structure and the results of our research has been published recently in two papers [10, 11]. The first paper [10] deals with the nondiagonal effective potentials which are the driving terms for all possible rearrangements of the three particles in $2 \rightarrow 2$ processes. Throughout the investigation it is assumed that all Coulomb potentials are repulsive, i.e., that the charges of all three particles are of the same sign. The new result is that the singularity in the momentum-transfer plane, which is the leading and, there- fore, the most dangerous one, is an integrable branch point located off the energy shell. Hence, it can never coincide, for values of the momenta in the integration region, with the pole of the effective free propagator. Consequently, the leading singularities of the nondiagonal kernels are integrable.

The second paper [11] deals with the singularity structure of the diagonal kernels. There it has be shown that, if the charges of all three particles are of the same sign, nonintegrable singularities appear only on the energy shell, and coincide below the breakup threshold with those considered by Veselova [6]. They can, however, be explicitly singled out and inverted as has been done by Alt and Sandhas [8]. Moreover, the off-the-energy-shell singularities of the diagonal kernels turn out to be integrable. These imply that after a few iterations the (suitably modified) effective-two-body AGS equations become integral equations with compact kernels.

The results of our investigation provide the proof that momentum space three-body integral equations in the form of effective two-body AGS equations can be used with confidence to calculate all possible arrangement (i.e., $2 \to 2$) amplitudes below and above the three-body threshold, provided the charges of all three particles are of the same sign. It is worth mentioning that from the proofs we presented also follows that, as soon as charges with opposite signs are involved, the kernels do, indeed, develop severe singularities which preclude application of standard methods of integral equations theory. We note also that the results obtained so far do not yet constitute a proof of compactness of the kernels of the integral equations of the Faddeev [1] or AGS [9] type for breakup processes $2 \to 3$. One obvious consequence is that application of methods which aim at directly solving these integral equations for $2 \to 3$ processes would (as yet) be without mathematical justification. Hence, it is of great importance to continue these investigations for the effective potentials occurring in the (integral) equations for $2 \to 3$ amplitudes.

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